# 2.34. More Duals

> This symmetry, or *duality* as it is called, which exists in principles and definitions, must also exist in all the formulas deduced from them as long as no principle or definition is introduced which would overthrow them. Hence a true formula may be deduced from another true formula by transforming it by the principle of duality…. In its application the *law of duality* makes it possible to replace two demonstrations by one.
>
> – Louis Couturat, **The Algebra of Logic** (1905)

To better appreciate the value of duality to the study of logic, we here survey some of the ways duality is woven through key concepts familiar from previous sections. This makes especially clear the ‘shortcuts’ duality offers: if features (and families) of sentences are guaranteed a logical ‘mirror image’, then a test for such a feature (or membership in a certain family) automatically brings with it a test for the dual of that feature (or family) as well.

**1. Duals of Sentences and Arguments**. It will hold in general that **the connective dual of a tautology is a contradiction**. That’s clear already from the True/False Swap, which always accompanies Connective Swap: the truth table for a tautology becomes the truth table for a contradiction, and vice versa. For instance, the connective dual of the tautology “(P ∨ ~P)” is the contradiction “(P ∧ ~P)”.

| | |
|---|---|
| **True** | **False** |
| **Disjunction** | **Conjunction** |
| **Tautology** | **Contradiction** |

**Negation**

And from that it follows that any sentence which is **neither a tautology nor** a **contradiction** has a dual sentence which is likewise neither.

Next, note that **a sentence in Disjunctive Normal Form has a sentence in Conjunctive Normal Form as its connective dual**, and any sentence in **neither DNF nor CNF** has a connective dual which is likewise in neither. So whereas “(~P ∧ Q) ∨ (P ∧ ~Q)” is in DNF, its connective dual “(~P ∨ Q) ∧ (P ∨ ~Q)” is in CNF.

| **True** | **False** |
|---|---|
| **Disjunction** | **Conjunction** |
| **Tautology** | **Contradiction** |
| **DNF Sentence** | **CNF Sentence** |

**Negation**

That observation illustrates the previous point about duality. For we saw that a DNF sentence, each of whose ‘cells’ (basic conjunctions) contains a sentence letter and its negation, is a contradiction.[1] “(~P ∧ P) ∨ (Q ∧ ~Q)”, for instance, is a DNF contradiction. And its connective dual is the CNF sentence “(~P ∨ P) ∧ (Q ∨ ~Q)” – which, having a sentence letter and its negation in each of its ‘cells’ (basic disjunctions), is a tautology. Once again: the dual of a contradiction is a tautology.

Concerning **validity**, note first: it is **not** true in general that if Sentence 2 follows validly from Sentence 1, then the connective dual of Sentence 2 follows validly from the dual of Sentence 1. For example, “~P” follows validly from “~(P ∨ Q)”; but the connective dual of “~P” (just “~P” again) doesn’t follow validly from the connective dual of “~(P ∨ Q)” – namely “~(P ∧ Q)”.

| **VALID** | **INVALID** |
|---|---|
| ~(P ∨ Q) | ~(P ∧ Q) |
| ∴ ~P | ∴ ~P |

[1] In 2.29.

But if **premise and conclusion switch places** in the dual argument, validity is preserved under (connective and semantic) duality: **if Sentence 2 follows validly from Sentence 1, the dual of Sentence 1 follows from the dual of Sentence 2**. For instance, "$\sim(P \wedge Q)$" follows validly from "$\sim P$".

| **VALID** | **VALID** |
|---|---|
| $\sim(P \vee Q)$ | $\sim P$ |
| $\therefore \sim P$ | $\therefore \sim(P \wedge Q)$ |

And we can see that this is bound to hold. For to say that Sentence 2 follows validly from Sentence 1 – that is, that the argument **Sentence 1 ∴ Sentence 2** is valid – is just to say that there's no validity counterexample, **no valuation where Sentence 1 is true but Sentence 2 is false**. But under the True/False Swap that claim becomes: there's no valuation where Dual of Sentence 1 is false but Dual of Sentence 2 is true.[2] That means: there's no validity counterexample for the argument **Dual of Sentence 2 ∴ Dual of Sentence 1**.[3]

> **If Sentence 2 follows validly from Sentence 1, then the (connective or semantic) dual of Sentence 1 follows validly from the dual of Sentence 2.**

| **VALID** | **VALID** |
|---|---|
| Sentence 1 | Dual of Sentence 2 |
| ∴ Sentence 2 | ∴ Dual of Sentence 1 |

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[2] Since the argument here is semantic, concerning which situations make a sentence true or false, the point is most obvious concerning **semantic** duals of premise and conclusion. But since this point holds for any semantic dual – they all, by definition, being logically equivalent – it holds, in particular, for the connective dual of those sentences.

[3] Borrowing an argument from (Quine 1959: 62).

That provides, in particular, a definition of **connective dual of an argument**.

> The **connective dual of an argument** is the result of (i) switching the conclusion and the premise(s), then (ii) replacing each sentence with its connective dual. (If the argument has more than one premise, these premises are conjoined together before Step (i).)

So the connective dual of argument "$\sim(P \vee Q) \therefore \sim P$" is "$\sim P \therefore \sim(P \wedge Q)$"; and the dual of the argument "$(P \vee Q)\ .\ \sim P \therefore Q$" is "$Q \therefore ((P \wedge Q) \vee \sim P)$".

From that point about dual arguments, it's easy to see that **logical equivalence is preserved under (semantic or connective) duality**. For two sentences are logically equivalent if (and only if) each follows validly from the other.[4]

> **If two sentences are logically equivalent, their duals are logically equivalent.**

So, for example, from either half of DeMorgan's Law we can derive the other half by duality.

"$\sim(P \wedge Q)$" **is equivalent to** "$(\sim P \vee \sim Q)$"
"$\sim(P \vee Q)$" **is equivalent to** "$(\sim P \wedge \sim Q)$"

[4] As noted in 2.20 and 2.25.

**2. Duals of Valuations and Counterexamples**. The points about argument validity carry over to argument invalidity. Most obviously: if an argument is invalid, its connective dual argument is invalid as well.[5] For example: since the argument P $\therefore$ (P $\wedge$ Q) is invalid, the argument (P $\vee$ Q) $\therefore$ P is likewise invalid.

| **INVALID** | **INVALID** |
|---|---|
| P | (P $\vee$ Q) |
| $\therefore$ (P $\wedge$ Q) | $\therefore$ P |

But there's more to be said here about invalid arguments and their duals. To begin, note that while we've used the True/False Swap on whole truth tables, we can apply it even to a single valuation: the semantic dual of a valuation is just the True/False Swap of that valuation.

**Semantic Dual of a Valuation**: the True/False Swap of that valuation

For instance, a valuation with "P" and "R" true and "Q" false will have as its semantic dual a valuation with "P" and "R" false and "Q" true.

**Valuation V**:

| **P** | **Q** | **R** |
|---|---|---|
| 1 | 0 | 1 |

**Dual of Valuation V**:

| **P** | **Q** | **R** |
|---|---|---|
| 0 | 1 | 0 |

Now an invalid argument has at least one **validity counterexample**. And if Argument A has valuation $V_A$ as a counterexample, the dual argument D(A) has as counterexample the dual of valuation $V_A$, D($V_A$).

[5] This follows from our earlier point about duals of valid arguments. For if there were some, on the contrary, an **invalid argument A** whose dual argument D(A) was nonetheless valid, then the dual of D(A) would also be valid. But thanks to the involutary nature of connective duality, the dual of D(A) is just argument A again. Hence **A would be valid**, contradicting the original assumption that A is invalid.

> **If an argument has a validity counterexample, then the connective dual of that argument has the semantic dual of that valuation as a validity counterexample.**

That’s clear from the True/False Swap: the valuation $V_A$ serving as validity counterexample for Argument A makes the (conjoined) premise of Argument A true and its conclusion false, hence (under the True/False Swap) makes the dual of that premise false and the dual of that conclusion true. So with premise and conclusion switched in the dual argument, this dual valuation makes the premise of the dual argument true and the conclusion false – a counterexample for the dual argument.

For instance, the counterexample for invalid argument “P ∴ (P ∧ Q)” is the second valuation, where **“P” is true and “Q” is false**. And the counterexample of the dual argument “(P ∨ Q) ∴ P” is the semantic dual of that valuation, where **“P” is false and “Q” is true**.

| **P** | Q | ∴ **(P ∧ Q)** |
|---|---|---|
| 1 | 1 | 1 |
| **1** | **0** | **0** |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

| P | Q | **(P ∨ Q)** | ∴ **P** |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| **0** | **1** | **1** | **0** |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |

**3. Duality and Negation**. It’s notable that many features of duality are shared with the semantics of **negation**. For the True/False Swap enacts the same semantic behavior as negation: replacing True with False and False with True. And **negation**, like the True/False Swap, is **semantically involutary**: the negation of the negation of a sentence is semantically equivalent to the original sentence.

The earlier observation about valid arguments and duality likewise holds for negations.

> **If Sentence 2 follows validly from Sentence 1, then the negation of Sentence 1 follows validly from the negation of Sentence 2.**

For example: "~P" follows validly from "~(P $\vee$ Q)"; and the negation of "~(P $\vee$ Q)" – equivalently, "(P $\vee$ Q)" – follows validly from the negation of "~P", semantically equivalently to "P".

Likewise, if two sentences are logically equivalent, their respective negations are also logically equivalent.

And just as the dual of a tautology is a contradiction (and vice versa), so the negation of a tautology is a contradiction (and vice versa). In fact we can intermix duality and negation here: the negation of the dual of a tautology is a tautology, as is the dual of the negation of a tautology.

Indeed, we could construct dual sentences by way of negation rather than through Connective Swap. For when the True/False Swap changes true to false and false to true in truth tables – for the sentence letters, and for the final, completed sentence – it's effecting the same change which negation makes. Hence we can achieve the same semantic result in truth tables by adding tildes – again, to each sentence letter, and to the whole sentence.

For instance, we know that the True/False Swap transforms the truth table for "(P $\vee$ Q)" into the truth table for "(P $\wedge$ Q)". But if we add a tilde to each sentence letter in "(P $\vee$ Q)" and a tilde to the whole sentence – yielding

"$\sim(\sim P \vee \sim Q)$" – we have a sentence with the same dual truth table picked out by the True/False Swap.

| P | Q | **(P ∨ Q)** | ~P | ~Q | (~P ∨ ~Q) | **~(~P ∨ ~Q)** | **(P ∧ Q)** |
|---|---|---|---|---|---|---|---|
| 1 | 1 | **1** | 0 | 0 | 0 | **1** | **1** |
| 1 | 0 | **1** | 0 | 1 | 1 | **0** | **0** |
| 0 | 1 | **1** | 1 | 0 | 1 | **0** | **0** |
| 0 | 0 | **0** | 1 | 1 | 1 | **0** | **0** |

Call this the "**Tilde Insertion**" method for constructing the dual of a sentence – yielding the "**tilde dual**" of a sentence.

> The **Tilde Dual** of a sentence is the result of placing a tilde before each sentence letter in the sentence, and before the entire sentence.

But while the dual offers a reasonable candidate for the dual of a sentence – in that there is just one tilde dual for each sentence, and that dual sentence always takes the correct dual truth table (the True/False Swap of the original sentence's truth table) – Tilde Insertion has less to recommend it than Connective Swap. For tilde duality, unlike connective duality, isn't involuntary in terms of connectives. For instance, the tilde dual of "$(P \vee Q)$" is "$\sim(\sim P \vee \sim Q)$". But the tilde dual of that sentence isn't "$(P \vee Q)$" again, but "$\sim\sim(\sim\sim P \vee \sim\sim Q)$". With Tilde Insertion, the dual of the dual isn't the original sentence.[6]

That said, Tilde Insertion has some useful applications when exploring sentence duality in particular languages. It will, for one, prove useful for calculating the semantic dual of a sentence in cases (such as in the languages of later chapters) where truth tables no longer suffice for sentence semantics.

[6] If we added to Tilde Insertion that we **remove double negations** and **apply (Inward) DeMorgan's Law** whenever possible, the Tilde Insertion procedure would usually yield the same results as the Connective Swap, and would be (to that degree) involutary. Call this the "**Simplified Tilde Dual**" dual of a sentence. Still, certain cases won't be involutary. For instance, the simplified tilde dual of "~~P" is "P". But the simplified tilde dual of "P" is "P" again, not "~~P".

As a second application, note that so long as a formal language contains the tilde (or its semantic equivalent), we can build a semantic dual of any sentence in that language, whether or not every connective finds its dual within that language. For instance, the formal language {∧, ~} contains lots of sentences that don't find their connective dual in the language. "(P ∧ Q)," for example, takes "(P ∨ Q)" as connective dual, but "(P ∨ Q)" isn't a sentence of language {∧, ~}. But since {∧, ~} contains the tilde we can construct the tilde dual of "(P ∧ Q)" – namely, "~(~P ∧ ~Q)" – guaranteed to take the same truth table the semantic dual of "(P ∧ Q)" would take.

That will be relevant in later discussion of the expressive power of formal languages – which truth tables a language can provide some matching sentence for. In the case of {∧, ~}, for instance, tilde duality alone guarantees that for every truth table matched by a {∧, ~} sentence, the semantic dual of that truth table also has a matching {∧, ~} sentence – namely, the tilde dual of the original sentence. In the jargon of (mathematics and) formal logic: any formal language containing the tilde (or its semantic equivalent) is **closed under semantic duality**, making that language a semantic **self-dual**.[7]

[7] The relevance of duality to the expressive power of languages is explored in 3.11.

**Summary: More Duals**

- The (connective or semantic) **dual of a tautology** is a contradiction (and vice versa).
- The connective **dual of a DNF sentence** is a CNF sentence (and vice versa)
- If Sentence S1 **validly entails** Sentence S2, then the (semantic or connective) dual of S2 validly entails the dual of S1.
- The connective **dual of an argument** is the result of (i) switching premise(s) and conclusion (conjoining premises if there’s more than one), and (ii) replacing each sentence by its connective dual.
- If two sentences are **logically equivalent**, their duals sentences are also logically equivalent.
- The **Tilde Insertion** procedure involves putting a tilde before each sentence letter in a sentence, and before the sentence as a whole. The result – the **tilde dual** of the sentence – is logically equivalent to the (connective or semantic) dual of that sentence.